

# On the Integrability of the Rabi Model

D. Braak\*

*EP VI and Center for Electronic Correlations and Magnetism,  
University of Augsburg, 86135 Augsburg, Germany*

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The exact spectrum of the Rabi hamiltonian is analytically found for arbitrary coupling strength and detuning. I present a criterion for integrability of quantum systems containing discrete degrees of freedom which shows that in this case a finite symmetry group may entail integrability, even without the presence of conserved charges beyond the hamiltonian itself. Moreover, I introduce and solve a natural generalization of the Rabi model which has no symmetries and is therefore probably the smallest non-integrable physical system.

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## I.

The Rabi (or single-mode spin-boson) model constitutes probably the simplest physical system beyond the harmonic oscillator. Introduced over 70 years ago [1], its applications range from quantum optics [2] and magnetic resonance to solid state [3] and molecular physics [4]. Very recently, it has gained a prominent role in novel fields of research such as cavity QED [5] and circuit QED [6]. It can be experimentally realized in Josephson junctions [7] or using trapped ions [8], in Cooper-pair boxes [9] and flux q-bits [10]. In this way, its complete theoretical understanding is mandatory for all feasible approaches to quantum computing [11]. Despite its old age and central importance, the Rabi model has not been exactly solved [3, 12, 13]. It shares with the other paradigm of quantum physics, the hydrogen atom, an infinite-dimensional state space but — in contrast to the latter — the spectrum and eigenfunctions of the Rabi model are known only by numerical diagonalization in a truncated, finite-dimensional Hilbert space. This is quite surprising, as the Rabi model has a smaller number of degrees of freedom than the hydrogen atom. In particular, a single degree of freedom  $\hat{x}$ , subject to a harmonic potential, couples to a quantum system with only two allowed states  $|\uparrow\rangle$  and  $|\downarrow\rangle$ . Therefore, it does not possess a classical limit: the quantum degree of freedom has a finite-dimensional Hilbert space and places the Rabi model in between the case of one and two (classical) degrees of freedom. The hamiltonian reads in units with  $\hbar = 1$ ,

$$H_R = \omega a^\dagger a + g \sigma_x (a + a^\dagger) + \Delta \sigma_z. \quad (1)$$

Here, the  $\sigma_{x,z}$  are Pauli matrices, describing the two-level system with level-splitting  $2\Delta$  and  $a$  ( $a^\dagger$ ) denote destruction (creation) operators of a single bosonic mode with frequency  $\omega$ . These two systems are coupled through a term proportional to  $g$ , which has different interpretations according to the experimental situation to model.

Although (1) describes the simplest of all physically sensible interacting quantum systems, it poses a serious obstruction to its analytical solution because of the apparent lack of a second conserved quantity besides the energy, which has led to the widespread opinion that it cannot be integrable [14–19]. To remedy this difficulty, Jaynes and Cummings proposed already in the sixties an approximation to (1) which does possess such a quantity [20]. Their hamiltonian reads

$$H_{JC} = \omega a^\dagger a + g(\sigma^+ a + \sigma^- a^\dagger) + \Delta \sigma_z. \quad (2)$$

with  $\sigma^\pm = (\sigma_x \pm i\sigma_y)/2$ .

Here, the operator  $C = a^\dagger a + \frac{1}{2}(\sigma_z + 1)$  commutes with  $H_{JC}$  and leads at once to the solvability of (2). The Jaynes-Cummings model is the so-called “rotating-wave” approximation to (1) and was justified because the conditions of near-resonance  $2\Delta \approx \omega$  and weak coupling  $g \ll \omega$  for such an approximation are realized in many experiments. The conservation of  $C$  means that the state space decomposes into an infinite sum of two-dimensional invariant subspaces, each labeled by the value of  $C = 0, 1, 2, \dots$ . Each eigenstate of (2) is then characterized by two quantum numbers, namely  $C$  and a two-valued index, for example  $+$  and  $-$ , denoting a basis vector in the two-dimensional subspace which belongs to  $C$ . Whereas the possible values of  $C$  form an unbounded set, corresponding to the quantization of a classical degree of freedom, the second quantum number can take only two values, reflecting the intrinsic quantum nature of the two-level system. The conserved quantity  $C$  generates a continuous  $U(1)$ -symmetry of the Jaynes-Cummings model (2) which is broken down to  $\mathbb{Z}_2$  in the Rabi model (1) due to the presence of the term  $a^\dagger \sigma^+ + a \sigma^-$ . This residual  $\mathbb{Z}_2$ -symmetry, usually called parity, leads to a decomposition of the state space into just two subspaces  $\mathcal{H}_\pm$ , each with infinite dimension. One would conclude that this symmetry cannot suffice to solve the model exactly — but in fact it does. Whereas a discrete symmetry is too weak to accommodate a classical (continuous) degree of freedom, it can do so with a quantum degree of freedom — like the two-level system in the Rabi model.

We observe a direct relation between the nature of the degree of freedom (continuous or discrete) and the sym-

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\*email: Daniel.Braak@physik.uni-augsburg.de

metry ( $U(1)$  versus  $\mathbb{Z}_2$ ), which can be used to “eliminate” it by fixing the corresponding irreducible representation.

Our main result is the following: The spectrum of (1) consists of two parts, the *regular* and the *exceptional* spectrum. Almost all eigenvalues are regular and given by the zeroes of the transcendental function  $G_{\pm}(x)$  in the variable  $x$ , which is defined through its power series in the coupling  $g$ :

$$G_{\pm}(x) = \sum_{n=0}^{\infty} K_n(x) \left[ 1 \mp \frac{\Delta}{x - n\omega} \right] \left( \frac{g}{\omega} \right)^n. \quad (3)$$

The coefficients  $K_n(x)$  are defined recursively,

$$nK_n = f_{n-1}(x)K_{n-1} - K_{n-2}, \quad (4)$$

with the initial condition  $K_0 = 1, K_1(x) = f_0(x)$  and

$$f_n(x) = \frac{2g}{\omega} + \frac{1}{2g} \left( n\omega - x + \frac{\Delta^2}{x - n\omega} \right). \quad (5)$$

The function  $G_{\pm}(x)$  is not analytic in  $x$  but has simple poles for  $x = 0, \omega, 2\omega, \dots$ , these poles are precisely the eigenvalues of the uncoupled bosonic mode. Then the regular energy spectrum of the Rabi model in each invariant subspace  $\mathcal{H}_{\pm}$  with parity  $\pm 1$  is given by the zeroes of  $G_{\pm}(x)$ : for all values  $x_n^{\pm}$  with  $G_{\pm}(x_n^{\pm}) = 0$ , the  $n$ -th eigenenergy with parity  $\pm 1$  reads  $E_n^{\pm} = x_n^{\pm} - g^2/\omega$ . The functions  $G_{\pm}(x)$  are plotted in Fig.1 for  $\Delta = 0.4$  and  $\omega = g = 1$  between  $x = -1$  and  $x = 5$ . Their zeroes determine the first six eigenenergies with parity  $+1$  and the first five (including the groundstate) with parity  $-1$ . For special values of model parameters  $g, \Delta$ , there

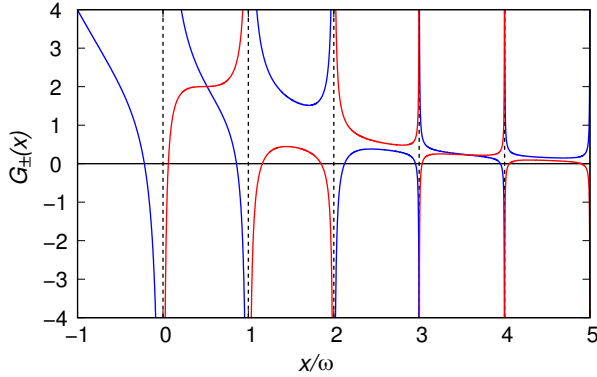


FIG. 1:  $G_+(x)$  (red) and  $G_-(x)$  (blue) in the interval  $[-1, 5]$  for  $g = 0.7$ ,  $\omega = 1$  and  $\Delta = 0.4$

are eigenvalues which do not correspond to zeroes of (3), these are the exceptional ones. All exceptional eigenvalues have the form  $E_n^e = n\omega - g^2/\omega$ , that is, they lie on one of the so-called baselines [21] and coincide with some point in the spectrum of the limiting case  $\Delta = 0$ , which corresponds in spin-boson language to zero hybridization. All exceptional eigenvalues are doubly degenerate with respect to parity. The necessary and sufficient condition

for an exceptional eigenvalue to lie on the  $n$ -th baseline reads,

$$K_n(n) = 0, \quad (6)$$

which furnishes a condition on the model parameters  $g$  and  $|\Delta|$ . As all the  $K_n(n)$  are independent polynomials in  $\Delta^2$ , there can exist at most two exceptional eigenvalues for given  $g, \Delta$ . These exceptional solutions to (1) have been known for a long time and were first discovered by Judd [21]. They occur when the pole of  $G_{\pm}(x)$  at  $x_n = n\omega$  is lifted because its numerator in (3) vanishes which happens only if (6) is satisfied.

The functional form of  $G_{\pm}(x)$  reads,

$$G_{\pm}(x) = G_{\pm}^0(x) + \sum_{n=0}^{\infty} \frac{h_n^{\pm}}{x - n\omega}, \quad (7)$$

where  $G_{\pm}^0(x)$  is entire in  $x$ . The position of the solutions to  $G_{\pm}(x) = 0$  is dictated by the pole-structure of  $G_{\pm}(x)$ , which leads to the conjecture that the number of eigenvalues in each interval  $[n\omega, (n+1)\omega]$  is restricted to be 0, 1, or 2. Moreover, for large energies  $x \gg g, \Delta$ , the entire part of  $G_{\pm}(x)$  can be approximated by

$$G_{\pm}^0(x) = \left( 1 \mp \frac{\Delta}{x} \right) \exp \left( -\frac{x}{2\omega} \right) \quad (8)$$

which is monotonous, always  $> 0$  and slowly varying on the scale set by  $\omega$ . This suggests that the relative position of the solution  $x_j(n)$  in the vicinity of the  $n$ -th baseline is fixed by the sign of  $h_n^{\pm}$  alone: If  $h_n^{\pm} > 0$  ( $h_n^{\pm} < 0$ ) then  $x_j(n)$  lies in the interval  $[(n-1/2)\omega, n\omega]$  ( $[n\omega, (n+1/2)\omega]$ ). It would also entail that an interval  $[n\omega, (n+1)\omega]$  with two roots of  $G_{\pm}(x) = 0$  can only be adjacent to an interval with one or zero roots; in the same way, an empty interval can never be adjacent to another empty interval. These conjectures about the distribution of the large eigenvalues, which can be confirmed numerically, lead to a fairly regular distribution of the energies and a violation of the Berry-Tabor criterion [16, 22]. Fig.2 shows the lowest part of the spectrum for  $\Delta = 0.4$ ,  $\omega = 1$  and  $g$  between 0 and 1. One may prove that there are no level crossings within each parity subspace, allowing the unique labeling of each state  $|\psi\rangle$  with a pair of two quantum numbers,  $|\psi\rangle = |n_0, n_1\rangle$ : The parity quantum number  $n_0$ , which takes the values  $+1$  and  $-1$ , and  $n_1 = 0, 1, 2, \dots$ , which denotes the  $n_1$ -th zero of  $G_{n_0}(x)$ . The exceptional solutions correspond to level crossings between  $\mathcal{H}_+$  and  $\mathcal{H}_-$ . This characterization of each eigenstate through two quantum numbers corresponding to the degrees of freedom of the system parallels the unique assignment of three quantum numbers  $n, l, m$  to the eigenstates of the hydrogen atom, reflecting the quantization of radial and angular degrees of freedom, a hallmark of integrability.

It seems therefore natural to call a quantum system integrable when such an assignment can be made — *independent* of the explicit determination of conserved quantities or even action variables, which is only possible if

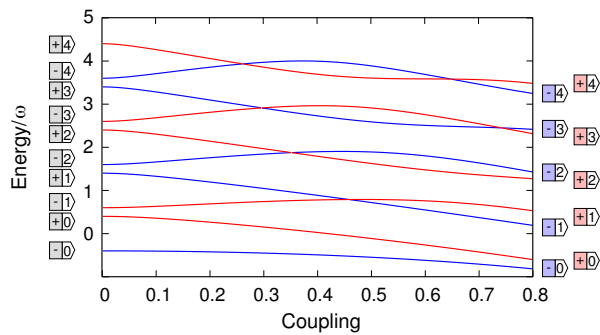


FIG. 2: Rabi spectrum for  $\Delta = 0.4$ ,  $\omega = 1$  and  $0 \leq g \leq 0.8$  in the spaces with positive (red) and negative (blue) parity. Within each space the states are labeled with ascending numbers  $0, 1, 2, \dots$ . This labeling does not change with  $g$  because no lines intersect within spaces of fixed parity. However, level crossings of states with different parity occur. The spectral graph is composed of two intersecting “ladders” of level lines, each corresponding to one parity subspace. This labeling is used on the right side of the figure. On the left side the states with  $g = 0$  are labeled by the uncoupled degrees of freedom, i.e. in  $|\pm, n\rangle$ ,  $+/ -$  corresponds to the two-level system and  $n = 0, 1, 2, \dots$  to the eigenstates of the bosonic mode.

the system under consideration has an integrable classical limit in the sense of Liouville [24]. Without making the assumption of a classical limit, our criterion reads,

*Criterion of quantum integrability:* If each eigenstate of a quantum system with  $f_1$  discrete and  $f_2$  continuous degrees of freedom can be *uniquely* labeled by  $f_1 + f_2 = f$  quantum numbers  $\{d_1, \dots, d_{f_1}, c_1, \dots, c_{f_2}\}$ , such that the  $d_j$  can take on  $\dim(\mathcal{H}_j)$  different values, where  $\mathcal{H}_j$  is the state space of the  $j$ -th discrete degree of freedom and the  $c_k$  range from zero to infinity, then this system is quantum integrable.

The criterion does not presuppose the existence of a family of commuting operators whose different spectra are associated with the  $\{d_j, c_k\}$  but provides a condition on the spectral graph of the system, that is, the spectrum as function of a parameter, typically one of the coupling constants. Without such a deformation parameter (which must conserve integrability) the association of more than one quantum number to the levels of a non-degenerate spectrum is ill-defined and would be restricted either to models solvable via Bethe ansatz [25] or systems with integrable classical limit. As is well-known, the Berry-Tabor criterion [22] relies precisely on the existence of this limit, that is, it applies only to the correspondence limit of large quantum numbers and presumes the validity of the semi-classical quantization scheme. The criterion proposed here is not restricted to the upper part of the spectrum or the existence of a classical limit. Moreover, it can be given a phenomenological formulation using the spectral graph: If the system with coupling  $g$  has  $f > 1$  discrete or continuous degrees of freedom, each

eigenstate can be written as  $|n_1, \dots, n_f; g\rangle$ . Upon varying the coupling  $g$ , each of these states defines an energy level  $E(n_1, \dots, n_f; g)$  as function of  $g$ . The totality of states for all sets  $\{n_1, \dots, n_f\}$  forms an  $f$ -dimensional manifold of spectral lines, which will typically intersect each other when energies corresponding to different invariant subspaces become accidentally degenerate at special values of the coupling. In a fully nondegenerate spectrum, a single integer quantum number which assigns  $0, 1, 2, \dots$  to the eigenstates with ascending energies would suffice for a unique description of each state — but whenever a level crossing occurs, this is no longer possible. A second quantum number is necessary to discern the energetically degenerate states. The crossing is *accidental* in the sense that no new symmetry appearing at the given value of the coupling parameter is responsible for the degeneracy, which will concern typically only two states in the spectrum. It is merely the fact that for these states, belonging to dynamically decoupled subspaces, the energy dependence on the coupling coincides. The second quantum number labels the invariant subspaces and can be used to discern the energetically degenerate states at the crossing point. According to the proposed criterion, Integrability is equivalent to the existence of  $f$  numbers to classify energy levels uniquely, if the system has  $f$  continuous or discrete degrees of freedom. It should be emphasized that these “quantum numbers” are a more general concept than the radial and angular quantum numbers known from atomic physics — they need not correspond to physical quantities (actions) quantized in integer multiples of  $\hbar$  and have nothing to do with the Bohr-Sommerfeld semi-classical quantization rules. They are only defined with respect to the unique description of eigenstates; the integrable systems differ from the non-integrable ones because they allow for a “fine-grained” description through an  $f$ -dimensional vectorial label, whereas the latter have a one-dimensional label corresponding to energy as the only conserved quantity. In the Rabi model we have  $f_1 = f_2 = 1$  and degeneracies takes place between levels of states having different parity, whereas within the parity subspaces no level crossings occur. The spectral graph consists of two “ladders” of level lines  $|n_0, m\rangle$  for  $m = 0, 1, 2, \dots$ . Each ladder corresponds to an invariant subspace of the  $\mathbb{Z}_2$ -symmetry characterized by  $n_0 = \pm$ , the parity eigenvalue. Within each subspace the system corresponds to a single continuous degree of freedom and is therefore integrable by definition. The global label (valid for all values of the coupling  $g$ ) is two-dimensional as  $f = f_1 + f_2 = 2$ ; the Rabi model belongs therefore to the class of integrable systems. It may be, however, that the symmetry is even stronger than necessary to achieve integrability, analogous to the classically “superintegrable” systems [26]. The Jaynes-Cummings model is an example for this case. Here, the continuous  $U(1)$ -symmetry leads to a further decomposition of the subspaces with fixed parity into a direct sum of two-dimensional invariant spaces labeled by the unbounded quantum number  $C = 0, 1, 2, \dots$ ; even (odd)

values of  $C$  correspond to odd (even) parity. The larger number of dynamically decoupled state spaces entails a second possibility to label the states uniquely: through  $C$  and a two-valued index  $n_0 = \pm$ . Now, all levels with different  $C$  may intersect, leading not to just two but infinitely many ladders labeled by  $C$ , each having two rungs, labeled by  $n_0$ . In Fig.3 the four lowest levels of the Rabi model with positive parity (red lines) are compared with the corresponding levels of the Jaynes-Cummings model with  $C = 1, 3, 5$  (black lines). The enlarged symmetry of the latter leads to two level crossings which are not present in the Rabi model [27]. The appearance of in-

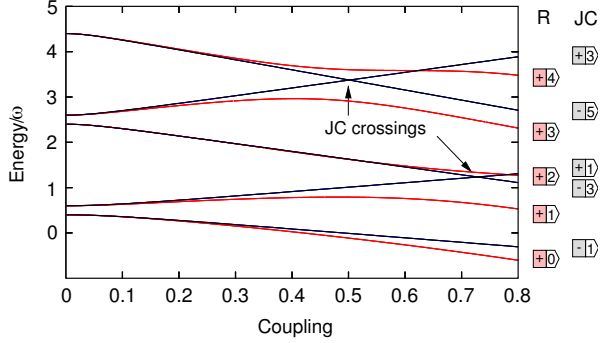


FIG. 3: The spectrum of the Jaynes-Cummings model (black) compared with the Rabi model (red) for  $\Delta = 0.4$  and even parity. The state labeling of the former has the form  $|\psi\rangle = |n_0, C\rangle$  with  $n_0 = \pm$ . Two accidental crossings occur between levels with  $C = 5$  and  $C = 3$  at  $g \approx 0.5$  and between  $C = 1$  and  $C = 3$  at  $g \approx 0.73$ .

tersecting ladders in the spectral graph can be detected without knowledge of the exact solution or the correct assignment of quantum numbers to the different levels, the only condition being a sufficient numerical resolution to discern degeneracies from narrow avoided crossings. This is a phenomenological virtue of the proposed criterion which could be used in computer experiments to test whether a given numerically solvable system possesses a hidden integrable structure. Although a large number of level crossings as a function of model parameters gives a strong hint to integrability, it is difficult to make the argument quantitative, because the number of intersecting ladders could become infinite already for two continuous degrees of freedom. For systems within the present class of one continuous and one discrete degree of freedom, however, its application is fairly obvious. The proposed criterion is sufficient but not necessary: in systems with factorized scattering matrices, a unique association of eigenstates with quantum numbers is possible which treats the discrete degrees of freedom differently from the present scheme, via the so-called nested Bethe ansatz [25]. But also this labeling leads to intersecting ladders as function of the coupling parameter.

On the other hand, the *absence* of any level crossings in the spectral graph is sufficient for non-integrability if the total number of degrees of freedom (continuous and

discrete) exceeds one: it means that the states can be classified only by energy, the single conserved quantity always present in hamiltonian systems, and no invariant subspaces exists. In the context of the Rabi model this case can be realized by breaking the  $\mathbb{Z}_2$ -symmetry. A possible generalization of (1) reads

$$H_\epsilon = \omega a^\dagger a + g \sigma_x (a + a^\dagger) + \epsilon \sigma_x + \Delta \sigma_z. \quad (9)$$

The term  $\epsilon \sigma_x$  breaks the parity symmetry which couples the bosonic mode and the two-level system. Physically it corresponds to a spontaneous transition of the two-level system which is not driven by the radiation field. The state space does not separate into two subspaces and indeed the spectral graph exhibits no level crossings at all if  $\epsilon$  is not a multiple of  $\omega/2$  [28]. This is shown in Fig.4. In

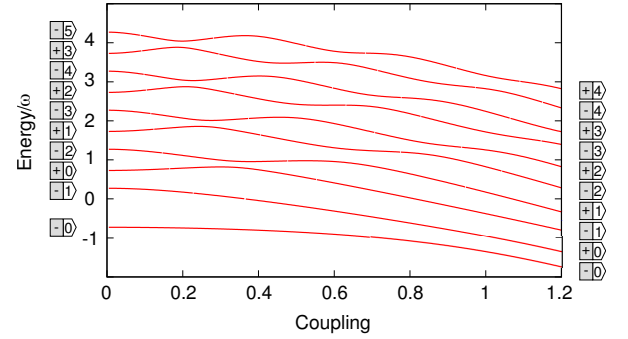


FIG. 4: The spectrum of the generalized Rabi model with broken  $\mathbb{Z}_2$ -symmetry and  $\Delta = 0.7, \epsilon = 0.2$ . The two-fold labeling of the states corresponds to the two integrable limits  $g = 0$  on the left and  $g \rightarrow \infty$  on the right. For finite  $g$  neither labeling classifies the states properly.

this situation, the eigenstates can be uniquely numbered as belonging to the  $n$ -th energy level in ascending order,  $|\psi\rangle = |n\rangle$ . We have only one quantum number, energy, corresponding to the sole conserved quantity. Because the number of degrees of freedom nevertheless exceeds one, this model must be considered non-integrable. Interestingly, (9) is still exactly solvable, although it does not possess any symmetry. Define the functions

$$R^\pm(x) = \sum_{n=0}^{\infty} K_n^\pm(x) \left(\frac{g}{\omega}\right)^n \quad (10)$$

and

$$\bar{R}^\pm(x) = \sum_{n=0}^{\infty} \frac{K_n^\pm(x)}{x - n\omega \pm \epsilon} \left(\frac{g}{\omega}\right)^n. \quad (11)$$

The  $K_n^\pm(x)$  are again recursively defined,

$$nK_n^\pm = f_{n-1}^\pm(x)K_{n-1}^\pm - K_{n-2}^\pm, \quad (12)$$

with the initial condition  $K_0^\pm = 1, K_1^\pm(x) = f_0^\pm(x)$  and

$$f_n^\pm(x) = \frac{2g}{\omega} + \frac{1}{2g} \left( n\omega - x \pm \epsilon + \frac{\Delta^2}{x - n\omega \pm \epsilon} \right). \quad (13)$$

The  $n$ -th eigenvalue  $E_n$  of (9) is given by the  $n$ -th zero  $x_n$  of

$$G_\epsilon(x) = \Delta^2 \bar{R}^+(x) \bar{R}^-(x) - R^+(x) R^-(x) \quad (14)$$

through  $E_n = x_n - g^2/\omega$ . The fact that  $H_\epsilon$  can be diagonalized analytically although not even a discrete symmetry is present signifies that integrability and solvability are not equivalent in the realm of quantum physics. In contrast to classical mechanics, non-integrable quantum systems with exact solutions exist.

## II. CONCLUSIONS

We have seen that the Rabi model is integrable contrary to common belief. The exact spectrum is given by the zeroes of a well-defined transcendental function  $G_\pm(x)$  which converges everywhere except at the set of points  $x \in \{n\omega | n = 0, 1, 2, \dots\}$ , where it has simple poles. The previously found exact eigenvalues for special values of the parameters  $g, \Delta$  are always two-fold degenerate and correspond to the merging of a zero with a pole of  $G_\pm(x)$  at  $x = n\omega$ . The eigenfunctions can be given explicitly as well (see appendix). The solution of the Rabi model appears in this way to be of comparable complexity with the well-known solution of the one-dimensional potential well, although it is a strongly interacting, fully quantum mechanical model. Its generalization with broken  $\mathbb{Z}_2$ -symmetry is non-integrable according to our criterion on integrability, which is tailored to the present class of systems with more than one but less than two degrees of freedom - nevertheless it is exactly solvable and describes probably the smallest non-integrable physical system. Whether exact solutions are possible for non-integrable quantum models with two or more continuous degrees of freedom is under current investigation.

## III. APPENDIX

To derive the result (3)-(5) for  $G_\pm(x)$  we use the representation of bosonic creation and annihilation operators in the Bargmann space of analytical functions in a complex variable  $z$  [23],

$$a \rightarrow \frac{\partial}{\partial z}, \quad a^\dagger \rightarrow z \quad (15)$$

Then, after applying a Fulton-Gouterman transformation and setting  $\omega = 1$ , the time-independent Schrödinger equation in the subspace  $\mathcal{H}_+$  with even parity reads

$$z \frac{d}{dz} \psi(z) + g \left( \frac{d}{dz} + z \right) \psi(z) = E \psi(z) - \Delta \psi(-z), \quad (16)$$

which is a *functional* differential equation in  $z$ . The solution  $\psi(z)$  is analytic in the whole complex plane if  $E$  belongs to the spectrum of  $H_R$ . With the notation

$\psi(z) = \phi_1(z)$ ,  $\psi(-z) = \phi_2(z)$ , one obtains a coupled system of first-order equations,

$$\begin{aligned} (z+g) \frac{d}{dz} \phi_1(z) + (gz-E) \phi_1(z) + \Delta \phi_2(z) &= 0 \\ (z-g) \frac{d}{dz} \phi_2(z) - (gz+E) \phi_2(z) + \Delta \phi_1(z) &= 0. \end{aligned} \quad (17)$$

With  $y = z + g$ ,  $x = E + g^2$ ,  $\phi_{1,2} = e^{-gy+g^2} \bar{\psi}_{1,2}$ , it follows,

$$y \frac{d}{dy} \bar{\psi}_1 = x \bar{\psi}_1 - \Delta \bar{\psi}_2 \quad (18)$$

$$(y-2g) \frac{d}{dy} \bar{\psi}_2 = (x-4g^2+2gy) \bar{\psi}_2 - \Delta \bar{\psi}_1. \quad (19)$$

$\bar{\psi}_2(y)$  is expanded into a power series in  $y$ ,

$$\bar{\psi}_2(y) = \sum_{n=-\infty}^{\infty} K_n(x) y^n \quad (20)$$

and from (18) one obtains for  $\bar{\psi}_1(y)$ ,

$$\bar{\psi}_1(y) = \sum_{n=-\infty}^{\infty} K_n(x) \frac{\Delta}{x-n} y^n. \quad (21)$$

Eq.(19) is then equivalent with the recurrence (4) and (5). For  $\bar{\psi}_{1,2}(y)$  to be analytic in  $y$  (i.e. at the point  $z = -g$ ) it is necessary that  $K_n(x) = 0$  for all  $n < 0$ . If the coefficient  $K_0$  is set to 1, this entails  $K_1(x) = f_0(x)$ , which fixes the initial conditions of (4). However, the power series in  $y$ , though analytic at  $z = -g$ , has the finite radius of convergence  $R = 2g$ , which can be deduced from the asymptotic value  $1/(2g)$  of  $f_{n-1}(x)/n$  for  $n \rightarrow \infty$ . Therefore,  $\bar{\psi}_2(z+g)$  will develop a branch-cut at  $z = g$ , if the parameter  $x$  does not belong to the discrete subset of  $\mathbb{R}$  which determines the spectrum of  $H_R$  in  $\mathcal{H}_+$ . But this condition on  $x$  follows easily from the fact that we have *two* representations for  $\psi(z)$  in  $\mathcal{H}_+$ , one constructed via  $\phi_2(z)$  and the other with  $\phi_1(z)$ ,

$$\psi(z) = \phi_2(-z) = e^{gz} \sum_{n=0}^{\infty} K_n(x) (-z+g)^n \quad (22)$$

$$\psi(z) = \phi_1(z) = e^{-gz} \sum_{n=0}^{\infty} K_n(x) \Delta \frac{(z+g)^n}{x-n}. \quad (23)$$

The expansion in powers of  $z$  is analytic at  $z = g$  in (22) and analytic at  $z = -g$  in (23). Only if both expansions are analytic at  $z = \pm g$  they may coincide everywhere and represent the same  $\psi(z)$  for all  $z \in \mathbb{C}$ . Therefore

$$G_+(x; z) = \phi_2(-z) - \phi_1(z) = 0 \quad \text{for all } z \in \mathbb{C} \quad (24)$$

if and only if  $x = E + g^2$  corresponds to a point in the spectrum of  $H_R$ . Because  $x$  is the only variable in  $G_+(x; z)$  besides  $z$ , it suffices to solve  $G_+(x; z) = 0$  for some arbitrarily chosen  $z$ . However, the function  $G_+(x; z)$  is only well-defined in  $x$  via its expansion in powers of  $z$  within the joint radius of convergence of (22)

and (23), which restricts the absolute value of  $z$  to be less than  $g$ . If  $x_0$  solves  $G_+(x; z) = 0$  for one such  $z$ , it will solve (24) as well and the eigenfunction  $\psi(z; E)$  for the energy  $E = x_0 - g^2$  possesses the two alternative series expansions (22, 23) which converge for  $x = x_0$  in the whole  $z$ -plane and are uniquely determined by  $x_0$ . It follows that all regular energy levels of  $H_R$  with positive parity are non-degenerate. Setting now  $z = 0$ , we obtain  $G_+(x) = G_+(x; 0)$ . This argument for  $\mathcal{H}_+$  carries over to  $\mathcal{H}_-$ , the subspace with negative parity, by replacing  $\Delta$  with  $-\Delta$ , which completes the derivation of (3). An exceptional solution occurs when a zero of  $G_\pm(x)$  merges with a pole at  $x = n\omega$ . As then  $\phi_2(-z) \neq \phi_1(z)$ , this point corresponds to a two-dimensional representation of  $\mathbb{Z}_2$  and the eigenvalue is two-fold degenerate.

There have been early attempts [29–31] to obtain an equation for the eigenvalue  $E(x)$  without the  $\mathbb{Z}_2$ -symmetry underlying the functional equation (16), as (17) arises after a rotation in spin space directly from (1). Then, of course, no functional relation between  $\phi_1$  and  $\phi_2$  can be used. Schweber has found the recurrence (4, 5) for the coefficients  $K_n(x)$  in

$$\bar{\psi}_2(y) = \sum_{n=0}^{\infty} K_n(x) y^n \quad (25)$$

and noted that the solution belongs only to the Hilbert space if a branchcut at  $y = 2g$  is avoided by the correct choice of  $x$ . One may formulate the problem in terms of the two kinds of solutions to the three-term recurrence (4) with initial condition  $K_0 = 1$ . The *dominant* solutions [32] all have the asymptotics

$$\lim_{n \rightarrow \infty} \frac{K_n(x)}{K_{n-1}(x)} = \frac{1}{2g} = \frac{1}{R}, \quad (26)$$

with finite radius of convergence  $R = 2g$  of (25) whereas the *minimal* solution has  $\lim_{n \rightarrow \infty} K_n(x)/K_{n-1}(x) = 0$ , which entails an infinite radius of convergence of (25), as mandated by analyticity. This imposes a second constraint on the initial condition for (4) besides

$$K_0 = 1, \quad K_1(x) = f_0(x), \quad (27)$$

i.e. on the allowed value of  $K_1$ , because the minimal solution is unique, whereas the dominant solutions are only determined modulo a multiple of the minimal one. This second constraint fixes  $x$  and therefore the spectrum. The procedure is then the following:

- 1.) Choose an arbitrary value  $x \in \mathbb{R}$ .
  - 2.) Compute numerically the minimal solution to (4) with  $K_0 = 1$ ,  $\{K_n^{\min}(x), n = 1, 2, \dots\}$ . The minimal solution depends on the  $f_n(x)$  with  $n \geq 1$ .
  - 3.) Compare  $K_1^{\min}(x)$  with  $f_0(x)$ .
- If both coincide,  $x$  will correspond to the eigenenergy  $E = x - g^2$ .

In [29, 30] it is claimed that a formally exact solution of the eigenvalue problem may be thus obtained in the following way: Define  $V_n = K_n/K_{n-1}$  for  $n = 1, 2, \dots$

Then (4) with initial condition (27) is equivalent to the two-term nonlinear recurrence relation

$$V_n = \frac{f_{n-1}(x)}{n} - \frac{1}{nV_{n-1}}, \quad n \geq 2 \quad (28)$$

with initial condition  $V_1 = f_0(x)$ , if none of the  $K_n$  vanishes. But (28) can also be written as

$$V_n = \frac{1}{f_n(x) - (n+1)V_{n+1}} \quad (29)$$

which determines  $V_n$  “from above” instead by the initial condition  $V_1 = f_0(x)$  together with (28). One could write  $V_1$  as a continued fraction,

$$V_1(x) = \frac{1}{f_1(x) - \frac{2}{f_2(x) - \frac{3}{f_3(x) - \dots}}} \quad (30)$$

Because the r.h.s. of (30) depends only on the set  $f_1(x), f_2(x), \dots$ , the formal equation

$$f_0(x) = 1|f_1(x) - 2|f_2(x) - 3|f_3(x) - \dots \quad (31)$$

seems to provide the sought condition on  $x$ , a transcendental equation, whose infinitely many roots determine the spectrum of  $H_R$ . This argument, however, is fallacious, because (31) is only a reformulation of the recurrence relation (28) with initial condition  $V_1(x) = f_0(x)$  and therefore valid for *all*  $x$ , regardless, whether they lie in the spectrum or not. Indeed, one may rewrite (31) as the finite continued fraction

$$f_0(x) = 1|f_1(x) - 2|f_2(x) - \dots (n-1)|f_{n-1}(x) - nV_n(x), \quad (32)$$

where  $V_n(x)$  depends on  $f_0(x)$  via (28) and amounts then simply to

$$f_0(x) = f_0(x). \quad (33)$$

Only if the r.h.s of (31) is *approximated* in one way or the other, it may give a non-trivial equation for  $x$ , whereas its *exact* value is tantamount to (33). In other words, the continued fraction in (30, 31) has no well-defined value without specifying the limiting behavior of the  $V_n$  for  $n \rightarrow \infty$ . It will converge both for the dominant solutions with  $V_n \rightarrow 1/2g$  as well as for the sought minimal solution with  $V_n \rightarrow 0$ , the difference being that in the latter case the resulting  $V_1$  will not depend strongly on the cut-off procedure, i.e. setting  $V_N = \varepsilon$  with  $\varepsilon \approx 0$  for some sufficiently large  $N$ , whereas the choice of  $N$  and  $\varepsilon$  has a strong impact on  $V_1$ , if one cuts off the continued fraction with  $V_N = \varepsilon + 1/2g$  [32]. This, of course, follows from the uniqueness of the minimal solution to (4).

Therefore (31) is equivalent to the numerical algorithm described above. Moreover, the cut-off at  $n = N$ , necessary to define its r.h.s., renders it equivalent with a diagonalization of  $H_R$  in a finite-dimensional Hilbert space, whose dimension depends on  $N$ .

Clearly, this problem with the representation of the spectral condition on  $x$  in terms of an infinite continued

fraction is a consequence of neglecting the  $\mathbb{Z}_2$ -symmetry which relates  $\phi_1(z)$  and  $\phi_2(z)$  in a nontrivial way and allows the definition of the function  $G_{\pm}(x)$ , whose zeroes determine the energy eigenvalues of the Rabi hamiltonian.

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